Exercise 29

Prove the formulas given in Table 6 for the derivatives of the following functions.

(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) csch^{-1}

(d) sech^{-1} (e) $\coth^{-1}$

Solution

Part (a)

Let

$$y = \cosh^{-1} x.$$

Then

$$\cosh y = x. \tag{1}$$

Take the derivative of both sides.

$$\frac{d}{dx}(\cosh y) = \frac{d}{dx}(x)$$

$$(\sinh y) \cdot \frac{dy}{dx} = 1$$

Solve for dy/dx.

$$\frac{dy}{dx} = \frac{1}{\sinh y}$$

Use the fact that $\cosh^2 y - \sinh^2 y = 1$.

$$\frac{dy}{dx} = \frac{1}{\pm\sqrt{\cosh^2 y - 1}}$$

$$= \frac{1}{\pm\sqrt{x^2 - 1}}$$

The hyperbolic cosine function is not one-to-one, so for an inverse function to exist, the argument has to be restricted to nonnegative values in equation (1). That is, $y \geq 0$, which means $\sinh y \geq 0$. Choose the plus sign.

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

Therefore,

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

Part (b)

Let

$$y = \tanh^{-1} x.$$

Then

$$\tanh y = x.$$

Take the derivative of both sides.

$$\frac{d}{dx}(\tanh y) = \frac{d}{dx}(x)$$

$$(\operatorname{sech}^2 y) \cdot \frac{dy}{dx} = 1$$

Solve for dy/dx.

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$$

Use the fact that $1 - \tanh^2 y = \operatorname{sech}^2 y$.

$$\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y}$$

$$= \frac{1}{1 - x^2}$$

Therefore,

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}.$$

Part (c)

Let

$$y = \operatorname{csch}^{-1} x.$$

Then

$$\operatorname{csch} y = x \quad \rightarrow \quad \frac{1}{\sinh y} = x \quad \rightarrow \quad \sinh y = \frac{1}{x}.$$

Take the derivative of both sides.

$$\frac{d}{dx}(\operatorname{csch} y) = \frac{d}{dx}(x)$$

$$(-\operatorname{csch} y \coth y) \cdot \frac{dy}{dx} = 1$$

Solve for dy/dx.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{\operatorname{csch} y \coth y} \\ &= -\frac{1}{\left(\frac{1}{\sinh y}\right)\left(\frac{\cosh y}{\sinh y}\right)} \\ &= -\frac{\sinh^2 y}{\cosh y} \end{aligned}$$

Use the fact that $\cosh^2 y - \sinh^2 y = 1$.

$$\frac{dy}{dx} = -\frac{\sinh^2 y}{\pm\sqrt{\sinh^2 y + 1}}$$

The hyperbolic cosine function is always positive, so choose the plus sign.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\sinh^2 y}{\sqrt{\sinh^2 y + 1}} \\ &= -\frac{\frac{1}{x^2}}{\sqrt{\frac{1}{x^2} + 1}} \\ &= -\frac{1}{x^2\sqrt{\frac{1+x^2}{x^2}}} \\ &= -\frac{1}{x^2\left(\frac{\sqrt{1+x^2}}{|x|}\right)} \end{aligned}$$

Therefore,

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}}.$$

Part (d)

Let

$$y = \operatorname{sech}^{-1} x.$$

Then

$$\operatorname{sech} y = x \quad \to \quad \frac{1}{\cosh y} = x \quad \to \quad \cosh y = \frac{1}{x}. \tag{2}$$

Take the derivative of both sides.

$$\frac{d}{dx}(\operatorname{sech} y) = \frac{d}{dx}(x)$$

$$(-\operatorname{sech} y \tanh y) \cdot \frac{dy}{dx} = 1$$

Solve for dy/dx.

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{\operatorname{sech} y \tanh y} \\ &= -\frac{1}{\left(\frac{1}{\cosh y}\right)\left(\frac{\sinh y}{\cosh y}\right)} \\ &= -\frac{\cosh^2 y}{\sinh y}\end{aligned}$$

Use the fact that $\cosh^2 y - \sinh^2 y = 1$.

$$\frac{dy}{dx} = -\frac{\cosh^2 y}{\pm\sqrt{\cosh^2 y - 1}}$$

The hyperbolic cosine function is not one-to-one, so for an inverse function to exist, the argument has to be restricted to nonnegative values in equation (2). That is, $y \geq 0$, which means $\sinh y \geq 0$. Choose the plus sign. Note also from equation (2) that since hyperbolic cosine is always positive, x is always positive as well.

$$\begin{aligned}\frac{dy}{dx} &= -\frac{\cosh^2 y}{\sqrt{\cosh^2 y - 1}} \\ &= -\frac{\frac{1}{x^2}}{\sqrt{\frac{1}{x^2} - 1}} \\ &= -\frac{1}{x^2\sqrt{\frac{1-x^2}{x^2}}} \\ &= -\frac{1}{x^2\left(\frac{\sqrt{1-x^2}}{x}\right)}\end{aligned}$$

Therefore,

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}.$$

Part (e)

Let

$$y = \coth^{-1} x.$$

Then

$$\coth y = x.$$

Take the derivative of both sides.

$$\frac{d}{dx}(\coth y) = \frac{d}{dx}(x)$$

$$(-\operatorname{csch}^2 y) \cdot \frac{dy}{dx} = 1$$

Solve for dy/dx.

$$\frac{dy}{dx} = -\frac{1}{\operatorname{csch}^2 y}$$

Use the fact that $\cosh^2 y - \sinh^2 y = 1$, which means $\frac{\cosh^2 y}{\sinh^2 y} - \frac{\sinh^2 y}{\sinh^2 y} = \frac{1}{\sinh^2 y}$, or $\coth^2 y - 1 = \operatorname{csch}^2 y$.

$$\frac{dy}{dx} = -\frac{1}{\coth^2 y - 1}$$

$$= -\frac{1}{x^2 - 1}$$

$$= \frac{1}{1 - x^2}$$

Therefore,

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}.$$